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Vector Spaces and Linear Inequalities

EUGENE LEVINE and HAROLD N. SHAPIRO

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Vector Spaces and Linear Inequalities

§1. Introduction.

The primary aim of this paper is to provide a simple and coherent development of the principal theorems concerning systems of linear inequalities in terms of the machinery of the theory of linear vector spaces. At its inception this development is based on a simple geometric separation theorem which is easily deduced from the standard theorem concerning the separation of closed convex bodies. This same geometric theorem also yields easily the existence of completely mixed strategies for the essential part of a rectangular game.

§2. A separation theorem for cones.

In order to prove the main result, it will be best to introduce notation and certain conventions.

1. All sets considered will lie in n-dimensional Euclidean space E^n.
2. If K is a set, then I(K) will denote the interior of K.
3. The first orthant of E^n is the set of all vectors $(x_1,\ldots,x_n)$ such that $x_i \geq 0$, $i = 1,\ldots,n$. The first orthant will be denoted by $\mathcal{O}$.
4. If C and K are two sets, then $(a,x) = b$ will be called a separating hyperplane of C and K, if $(a,x) \geq b$ for all $x \in C$ and $(a,x) \leq b$ for all $x \in K$. If C and K have a separating hyperplane then we say that C and K can be separated.

We will also assume the following well known theorem:

Theorem 1. If C and K are disjoint and convex, then C and K can be separated.

Corollary 1. If C and K are convex and intersect only in common boundary points, and if $I(K) \neq 0$, then C and K can be separated.

Proof: C and $I(K)$ can be separated, and hence the same hyperplane separate C and K.

We now present the main result of this section.

Theorem 2. Let C be a closed convex cone such that $C \cap \mathcal{O} = \vec{0}$. Then there exists a hyperplane $(a,x) = 0$ such that $(a,x) \leq 0$ for $x \in C$ and $(a,x) > 0$ for $x \in \mathcal{O}$, $x \neq \vec{0}$.

Proof: Let e^j $(j = 1,\ldots,n)$ be the j-th unit vector in E^n. Let $e^j(\varepsilon) = (-\varepsilon,-\varepsilon,\ldots,\underbrace{1}_{j},\ldots,-\varepsilon)$. We now denote the cone generated by $e^j(\varepsilon)$, $(j = 1,\ldots,n)$ as $\mathcal{O}(\varepsilon)$. We will now show that if $0 \leq \varepsilon < 1/n-1$, then $\mathcal{O}(\varepsilon) \supset \mathcal{O}$.

For $e^j(\varepsilon) = e^j - \varepsilon \sum_{k \neq j} e^k = (1+\varepsilon)e^j - \varepsilon \sum_{k=1}^{n} e^k$. Thus

$$\sum_k e^k(\varepsilon) = (1+\varepsilon)\sum_k e^k - \varepsilon n \sum_k e^k = \{1 - \varepsilon(n-1)\} \sum_k e^k ,$$

hence

$$(1+\varepsilon)e^j = e^j(\varepsilon) + \frac{\varepsilon}{1 - \varepsilon(n-1)} \sum_k e^k(\varepsilon) ,$$

or

$$e^j = \frac{e^j(\varepsilon)}{1+\varepsilon} + \frac{\varepsilon}{(1+\varepsilon)(1-\varepsilon(n-1))} \sum_k e^k .$$

Hence the e^j's are positive linear combinations of the $e^j(\varepsilon)$'s, i.e. $\mathcal{O}(\varepsilon) \supset \mathcal{O}$.

It also follows that if $1/n-1 > \varepsilon \geq \eta \geq 0$, then $\mathcal{O}(\varepsilon) \supset \mathcal{O}(\eta)$. For

$$e^j(\varepsilon) - e^j(\eta) = (\varepsilon-\eta)e^j - (\varepsilon-\eta)\sum_k e^k = -(\varepsilon-\eta)\sum_{k\neq j} e^k \quad .$$

Thus $e^j(\eta) = e^j(\varepsilon) + (\varepsilon-\eta)\sum_{k\neq j} e^k$. Thus $e^j(\eta)$ is a non-negative linear combination of vectors in $\mathcal{O}(\varepsilon)$, i.e. $\mathcal{O}(\varepsilon) \supset \mathcal{O}(\eta)$.

We now want to show that there exists $\varepsilon > 0$, such that $\mathcal{O}(\varepsilon) \cap C = \vec{0}$. Suppose this were not the case. Let $\varepsilon_i \to 0$ be a monotonic sequence where $1/n-1 > \varepsilon_i > 0$. Then $\mathcal{O}(\varepsilon_i) \cap C$ contains a vector $x_i \neq \vec{0}$. Thus $\mathcal{O}(\varepsilon_i) \cap C$ contains $\bar{x}_i = x_i/\|x_i\|$. But $\{\bar{x}_i\}$ is a bounded sequence hence it contains a converging subsequence $\{\bar{x}_{i_k}\} \to x^* \neq \vec{0}$. Thus $x^* \in C$ and $x^* \in \mathcal{O}(\varepsilon_i)$ for each i. This implies that $x^* \in \bigcap_{i=1}^{\infty} \mathcal{O}(\varepsilon_i) = \mathcal{O}$. Thus $x^* \in \mathcal{O} \cap C$ which is a contradiction. Hence there exists ε^* such that $0 < \varepsilon^* < 1/n-1$ and $C \cap \mathcal{O}(\varepsilon^*) = \vec{0}$.

Now C and $\mathcal{O}(\varepsilon^*)$ are convex, $C \cap \mathcal{O}(\varepsilon^*)$ contains only a common boundary point, and $\mathcal{O}(\varepsilon^*)$ contains an interior point. Thus there exists a hyperplane separating C and $\mathcal{O}(\varepsilon^*)$ which is the desired hyperplane.

<u>Theorem 3</u>. Let P be a polyhedron such that $P \cap \mathcal{O} = \vec{0}$. Then there exists a hyperplane separating P and $\mathcal{O}$ which intersects $\mathcal{O}$ only at the origin.

<u>Proof</u>: Let P be the convex hull of $p_1, \ldots, p_t$. Let P^* be the convex cone generated by $p_1, \ldots, p_t$. Then $P^* \supset P$ and P^* is closed. We now want to show that $P^* \cap \mathcal{O} = \vec{0}$. Suppose $u \in P^*$ and $u \in \mathcal{O}$ where $u \neq \vec{0}$. Then $u = \lambda_1 p_1 + \ldots + \lambda_t p_t$ where $\lambda_i \geq 0$ and $\lambda = \sum \lambda_i > 0$. Then it is clear that $\mu/\lambda \in \mathcal{O}$, $\mu/\lambda \in P$ and $\mu/\lambda \neq \vec{0}$ contradicting the hypothesis. Thus $P^* \cap \mathcal{O} = \vec{0}$. Then by Theorem 2, P^* and $\mathcal{O}$ have a separating hyperplane which meets $\mathcal{O}$ only at the origin. Clearly this hyperplane separates P and $\mathcal{O}$ as desired.

§3. Properties of vector spaces derived from the separation theorem.

In this section we will derive some properties of vector spaces, and we will show how each such property can be translated into a theorem on matrices.

Theorem 4. Let V be a vector subspace of E^n. Then either

1. $V \cap \mathcal{O} \neq \vec{0}$

or 2. $V^{\perp} \cap I(\mathcal{O}) \neq 0$.

Proof: We first note that V is a closed convex cone. Now negating (1) we have $V \cap \mathcal{O} = \vec{0}$. Thus by Theorem 2, there exists a separating hyperplane $(a,x) = 0$ such that $(a,v) \leq 0$ for $v \in V$ and $(a,s) > 0$ for $s \in \mathcal{O}$, $s \neq \vec{0}$. Now $v \in V$ implies $-v \in V$, thus $-(a,v) = (a,-v) \leq 0$, hence $(a,v) \geq 0$. Thus $(a,v) = 0$ for all $v \in V$. Thus $a \in V^{\perp}$. To show that $a \in I(\mathcal{O})$, we just note that $(a,e^i) > 0$ implies $a_i > 0$ for each i, thus we have the second alternative.

Definition. A vector $a = (a_1,\ldots,a_n)$ will be called positive if $a_i \geq 0$, $i = 1,\ldots,n$ and $\sum a_i > 0$. The vector is said to be strictly positive if $a_i > 0$ for $i = 1,\ldots,n$.

Corollary. (Steimke's Theorem) Let A be an $m \times n$ matrix. Then either

(1) there exists a vector v such that vA is positive,

or (2) there exists a strictly positive vector w such that $Aw = \vec{0}$.

Proof: Let $V = A'E^m$. Now if (1) does not hold, then $V \cap \mathcal{O} = \vec{0}$. Thus by Theorem 4, $V^{\perp} \cap I(\mathcal{O}) \neq 0$. Thus there exists

a strictly positive vector w such that $(w,v) = 0$ for all $v \in V$. Thus $(w, A'x) = 0$ for all $x \in E^m$. Thus $(Aw, x) = 0$ for all $x \in E^m$. Hence $Aw = \vec{0}$ which is the second alternative.

We now note the "dual" to Theorem 4, namely

Theorem 5. Let V be a vector subspace of E^n. Then either

1. $V \cap I(\sigma) \neq 0$

or 2. $V^{\perp} \cap \sigma \neq \vec{0}$.

Proof: Applying Theorem 4 to $V^{\perp}$ we have the result.

Corollary. Let A be an m n matrix. Either

1. There exists a vector v such that vA is strictly positive,

or 2. There exists a positive vector w such that $Aw = \vec{0}$.

Proof: Follow similarly to the previous corollary.

We now pursue some further theorems on vector spaces and linear inequalities which generalize the preceding results.

For this purpose, we will look upon E^n as being the product space $E^r \times E^s$, where E^r is the r-dimensional euclidean space of the first r cartesian coordinates of E^n, and E^s the corresponding space of the last $s = n-r$ coordinates. We may then look upon the first orthant of E^n, which we now denote by σ^n, as the product space $\sigma^r \times \sigma^s$. We will also use the following notation: If $u = (u_1, \ldots, u_n) \in E^n$, then $u^{(r)} = (u_1, \ldots, u_r)$ and $u^{(s)} = (u_{r+1}, \ldots, u_n)$. Furthermore, if $S \subset E^n$, then $S^{(r)} = \{u^{(r)} | u \in S\}$ and $S^{(s)} = \{u^{(s)} | u \in S\}$.

Theorem 6. Let V be a vector subspace of E^n. Let $E^n = E^r \times E^s$, $(r+s=n)$ and $\sigma^n = \sigma^r \times \sigma^s$. Then either

(1) $[V \cap \sigma^n]^{(r)} \neq \vec{0}^{(r)}$

or (2) $V^{\perp} \cap [I(\sigma^r) \times \sigma^s] \neq 0$.

Proof: The proof will proceed by induction on s. For $s = 0$, the statement reduces to Theorem 4. We now assume the theorem to be true for all s' such that $0 \leq s' < s$. Applying the induction hypothesis to $E^{r+1} \times E^{s-1}$, (where E^{r+1} is obtained by "adjoining" to E^r any one of the last s coordinates) we obtain either

$$(1)_{r+1}: \quad [V \cap \mathcal{O}^n]^{(r+1)} \neq \vec{0}^{(r+1)}$$

or $$(2)_{r+1}: \quad V^{\perp} \cap [I(\mathcal{O}^{r+1}) \times \mathcal{O}^{(s-1)}] \neq 0 \quad .$$

Suppose $(2)_{r+1}$ holds. Then noting that

$$I(\mathcal{O}^{(r+1)}) \times \mathcal{O}^{(s-1)} \subset I(\mathcal{O}^r) \times \mathcal{O}^s$$

we have

$$(2): \quad V^{\perp} \cap [I(\mathcal{O}^r) \times \mathcal{O}^s] \neq 0$$

which is the second alternative. Thus we may assume that $(1)_{r+1}$ holds.

Now if we negate (1), we have $[V \cap \mathcal{O}^n]^{(r)} = \vec{0}^{(r)}$. Then by $(1)_{r+1}$, there must exists $P_j \in V$, $(j = 1,\ldots,s)$ such that

$$P_j = (u_1,\ldots,u_r,u_{r+1},\ldots,u_{r+j},\ldots,u_{r+s}) \in \mathcal{O}^n$$

and $(u_1,\ldots,u_r,u_{r+j}) \neq \vec{0}^{(r+1)}$. But by negating (1), we have $(u_1,\ldots,u_r) = \vec{0}^{(r)}$. Thus $u_{r+j} > 0$. Let $P = \sum_{j=1}^{s} P_j$. Then $P \in V$, $P^{(r)} = \vec{0}^{(r)}$ and $P^{(s)}$ is strictly positive.

Now suppose $V^{(r)} \cap \mathcal{O}^{(r)} \neq \vec{0}^{(r)}$. (Note: In general $V^{(r)} \cap \mathcal{O}^{(r)} \neq [V \cap \mathcal{O}]^{(r)}$.) Then there exists $v \in V$ such that $v^{(r)} \in \mathcal{O}^{(r)}$ and $v^{(r)} \neq \vec{0}^{(r)}$. Clearly $v+\lambda P \in V$, and for sufficiently large λ, $v+\lambda P \in \mathcal{O}$. Then $\vec{0}^{(r)} \neq (v+\lambda P)^{(r)} \in [V \cap \mathcal{O}]^{(r)}$

which contradicts our assumption (namely the negation of (1)). Thus $V^{(r)} \cap \mathcal{O}^{(r)} = \vec{0}^{(r)}$. Hence, by Theorem 4, $(V^{(r)})^{\perp} \cap I(\mathcal{O}^{(r)}) \neq 0$. Thus there exists $Z^{(r)} \in (V^{(r)})^{\perp}$ such that $Z^{(r)}$ is strictly positive. Let Z be the vector obtained from $Z^{(r)}$ by adjoining s zeros to $Z^{(r)}$. Then clearly $Z \in V^{\perp}$ and $Z \in I(\mathcal{O}^{(r)}) \times \mathcal{O}^{(s)}$. Thus $V^{\perp} \cap [I(\mathcal{O}^{r}) \times \mathcal{O}^{s}] \neq 0$, and (by negating (1)) we have shown that (2) must hold.

<u>Theorem 7</u>. ("Dual" to Theorem 6) Let V be a vector subspace of E^n. Then either

(1) $[V^{\perp} \cap \mathcal{O}^{n}]^{(r)} \neq \vec{0}^{(r)}$

or (2) $V \cap [I(\mathcal{O}^{r}) \times \mathcal{O}^{s}] \neq 0$.

<u>Proof</u>: Apply Theorem 6 to $V^{\perp}$.

The following corollaries indicate how the above theorems translate into statements concerning systems of inequalities.

<u>Definition</u>. Let $x = (x_1, \ldots, x_n) \in E^n$. Then $x > 0$ will designate that x is strictly positive; $x \geq 0$ will designate that x is positive; and $x \geqq 0$ will designate that $x_i \geqq 0$, $i = 1, \ldots, n$.

<u>Corollary</u>. (to Theorem 6) Let A and B be matrices of dimensions $m \times r$ and $m \times s$ respectively. Then either

(1) there exists a vector u such that

$$A'u \geq 0 \quad \text{and} \quad B'u \geqq 0$$

or (2) there exists a vector $x > 0$ and a vector $y \geqq 0$ such that $Ax + By = \vec{0}$.

(Note: For $s = 0$, this is Steimke's Theorem.)

Proof: Let $C = (A,B)$ (hence C is of dimension $m \times (r+s)$) so that $C' = \binom{A'}{B'}$. Now $C'(E^m) = V \subset E^r \times E^s$. Applying Theorem 6 to V, we have

$$\text{either} \quad (1') \quad [V \cap \mathcal{O}^{(r+s)}]^{(r)} \neq \vec{0}^{(r)}$$

$$\text{or} \quad (2') \quad V^{\perp} \cap [I(\mathcal{O}^{r}) \times \mathcal{O}^{s}] \neq 0 \quad .$$

If (1') holds, then there exists $v \in V$ such that $v^{(r)} \geq 0$ and $v^{(s)} \geqq 0$. Now there is some $u \in E^m$ such that $C'u = v$, and for this u, $A'u = v^{(r)}$ and $B'u = v^{(s)}$ so that (1) follows.

If (2') holds, then there exists $v \in V^{\perp}$ such that $v^{(r)} > 0$ and $v^{(s)} \geqq 0$. Then for all $u \in E^m$, $0 = (C'u,v) = (u,Cv)$, so that $Cv = \vec{0}$. Thus

$$\vec{0} = Cv = (A,B)\binom{v^{(r)}}{v^{(s)}} = Av^{(r)} + Bv^{(s)}$$

and alternative (2) follows in this case.

Corollary. (to Theorem 7) (The "dual" of the previous corollary.) Let A and B be matrices of dimensions $m \times r$ and $m \times s$ respectively. Then either

(1) there exists a vector $x \geq 0$ and a vector $y \geqq 0$ such that $Ax + By = \vec{0}$

or (2) there exists $u \in E^m$ such that $A'u > 0$ and $B'u \geqq 0$.

Proof: Follows from the alternatives of Theorem 7.

§4. Derivation of Farkas' Theorem.

We now illustrate the application of the above inequalities by deriving Farkas' theorem from them.

Theorem 8. (Inhomogeneous form) Let B be an $m \times s$ matrix, let b be an s-dimensional (column) vector, and let x be an m-dimensional (column) vector. Let $\mathcal{P} = \{u \mid B'u-b \geqq 0\}$ (note

that $\mathcal{D}$ is a subset of E^m). If $\mathcal{D} \neq 0$, and if $u \in \mathcal{D}$ implies $(x',u) \geqq p$, then there exists $y \geqq 0$ such that

(1) $x = By$

<u>and</u> (2) $\bar{p} = (y',b)$ where $\bar{p} = \min\limits_{u \in \mathcal{D}} (x',u)$.

<u>Proof</u>: Let $x' = (x_1, x_2, \ldots, x_m)$ and $b' = (b_1, b_2, \ldots, b_s)$. Let

$$A = \begin{pmatrix} 0 & -x_1 \\ 0 & -x_2 \\ \vdots & \vdots \\ 0 & -x_m \\ 1 & \bar{p} \end{pmatrix} = \begin{pmatrix} 0_{m \times 1} & -x_{m \times 1} \\ 1_{1 \times 1} & \bar{p}_{1 \times 1} \end{pmatrix}$$

and let

$$\hat{B} = \begin{pmatrix} B_{m \times s} \\ -b'_{1 \times s} \end{pmatrix} .$$

Applying the corollary (to Theorem 7) to the matrices A and B, we have that

<u>either</u> (1') there exists a vector $\xi \geq 0$ and a vector $y \geqq 0$ such that $A\xi + \hat{B}y = \vec{0}$

<u>or</u> (2') there exists $u \in E^{m+1}$ such that $A'u > 0$ and $\hat{B}'u \geqq 0$.

We now show that (2') cannot hold. For suppose (2') held, then there exists $u = (u_1, \ldots, u_{m+1})$ such that $A'u > 0$ and $\hat{B}'u \geqq 0$. Let $u^{(m)} = (u_1, \ldots, u_m)$. Then $B'u^m - bu_{m+1} \geqq 0$. But $A'u > 0$, hence $u_{m+1} > 0$ and $-(x',u^m) + \bar{p}u_{m+1} > 0$. Let $\hat{u} = (u^m/u_{m+1})$. Then

$$B'\hat{u} - b \geqq 0 \quad \text{and} \quad \bar{p} > (x',\hat{u}) \quad .$$

Thus $\hat{u} \in \mathscr{F}$, hence $\bar{p} = \min_{u \in \mathscr{F}} (x',u) \leq (x',u) < \bar{p}$ which is a contradiction. Thus we conclude that (1') holds.

Thus there exists $\xi = (\xi_1, \xi_2) \geq 0$ and $y \geqq 0$ such that $A\xi + \hat{B}y = \vec{0}$, where $\xi_1 + \xi_2 > 0$. We may also assume that $\xi_1 + \xi_2 = 1$ (by dividing through by $\xi_1 + \xi_2$). Now

$$A\xi + \hat{B}y = \begin{pmatrix} 0 & -x \\ 1 & \bar{p} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} B \\ -b' \end{pmatrix} y = \begin{pmatrix} -x\xi_2 + By \\ \xi_1 + \bar{p}\xi_2 - b'y \end{pmatrix} = \vec{0}$$

hence $-x\xi_2 + By = \vec{0}$ and $\xi_1 + \bar{p}\xi_2 - b'y = 0$.

We now show that $\xi_2 \neq 0$. For if $\xi_2 = 0$, then $By = \vec{0}$ and $b'y = 1$. Then for any $u \in \mathscr{F}$, $0 = y'B'u \geqq y'b = 1$ which is a contradiction. Thus $\xi_2 > 0$.

Then $x = B(y/\xi_2)$ and $(\xi_1/\xi_2) + \bar{p} = b'(y/\xi_2)$. Let $z = (y/\xi_2)$ and let $u^* \in \mathscr{F}$ be such that $\bar{p} = (x', u^*)$. Then

$$\bar{p} = (x', u^*) = z'B'u^* \geq z'b = \bar{p} + \xi_1/\xi_2 \quad .$$

Thus $\xi_1 = 0$ and hence $\xi_2 = 1$. Therefore $x = By$ and $\bar{p} = (b'y) = (y', b)$.

<u>Corollary</u>. (Homogeneous form) Let B be an $m \times s$ matrix and let $x \in E^m$. If for all $u \in E^m$ such that $B'u \geqq 0$ we have $(x',u) \geqq 0$, then $x = By$ where $y \geqq 0$.

<u>Proof</u>: In Theorem 8, set $b = \vec{0}$, $p = 0$ and we have the result.

§5. An application to a dimension relationship.

As a further illustration of the separation theorem, we apply it to a fundamental dimension relationship in game theory. We state the dimension relationship in the following:

Theorem. Let Π_I and Π_{II} be the spaces of optimal strategies for a game matrix A. Let Σ_I and Σ_{II} be the smallest faces of the simplices of admissible strategies which contain Π_I and Π_{II} respectively. Then

$$D(\Sigma_I) - D(\Pi_I) = D(\Sigma_{II}) - D(\Pi_{II})$$

where D(S) denotes the dimension of a set S.

We will not give a proof of the full theorem (see [2]) but shall restrict ourselves to an essential feature of the proof. In order to do this we introduce:

Definition. A vector strategy is called completely mixed if every component is positive.

Then the essential feature of the dimension relationship is imbedded in:

Theorem 9. If A is a reduced game matrix, then there exists a completely mixed optimal strategy for each player.

Proof: Since A and A+cl have the same optimal strategies, we may assume that the value of the game v = 0. Let $R_1,\dots,R_m$ be the rows of A, and $C_1,\dots,C_n$ be the columns. Then $(x,C_j) = 0$ for all j where $x \in \Pi_I$ and $(y,R_i) = 0$ for all i where $y \in \Pi_{II}$. Thus if $x \in \Pi_I$, $\sum_{i=1}^{m} x_i R_i = \vec{0}$. Let P be the convex hull of $R_1,\dots,R_m$. Then $\vec{0} \in P$. Now consider any convex combination of $R_1,\dots,R_m$ such that $\sum_{i=1}^{m} x_i R_i \geqq \vec{0}$. This implies that $(x_1,\dots,x_m)$

is optimal, thus equality must hold. Hence $P \cap \mathfrak{S} = \vec{0}$. Thus by the separation theorem for polyhedra, there exists a hyperplane $(a,x) = 0$ such that $(a,p) \leq 0$ for $p \in P$ and $(a,s) > 0$ for $s \in \mathfrak{S}$, $s \neq \vec{0}$. Letting e_i be the i-th unit vector, we have $(a,e_i) > 0$, hence $a_i > 0$. Thus we may normalize to obtain a strategy vector a' which is completely mixed. But $(a',R_i) \leq 0$, hence $a' \in \Pi_{II}$, i.e. a' is optimal for the second player.

To find a completely mixed optimal strategy for the first player, we need only to apply the result to the transpose of A.

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